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MSc DISSERTATION

Causal Sets from Classical Sequential Growth Models

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Abstract

In the causal set program, the development of a dynamics is one of the most crucial issues. In this report, a candidate for a dynamics, sequential growth models, is studied with focus on the author's personal work. Causal set formalism is introduced using the more general mathematics of partial orders. Sequential growth models are investigated as a possible dynamics for causal sets. Among them, transitive percolation is particularly studied and several mathematical results are proved. Properties of causal sets are then observed in simulations, and evolution of these properties is particularly stressed. Finally, sequential growth models are studied as a part of a larger collection of models attributing probabilities to causal sets.

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Introduction

The argument between supporters of continuous and discrete approaches to physics is a long standing one. For those who refer discrete approaches ([1]), granularity might be the solution to the problems faced in modern physics such as infinities and singularities in Quantum Field Theory and General Relativity. In parallel, work on causal structure in general relativity showed that the geometry of a manifold, described by a Lorentzian metric, is given by the causal structure up to a conformal factor ([2]). These two ideas, discreteness and causality are the basis of Causal Sets. As the conformal factor can be seen as the volume function on the manifold, in causal sets discreteness enables replacing volume by the number of elements, simply by associating a volume to every element - hence the metric would be equivalent to a partial order on a discrete, locally finite set. This "causal set" approach could then incorporate general relativity, and may be the basis for describing quantum physics, thus solving the problem of quantum gravity.

Furthermore, assuming the validity of Lorentz invariance to any scale, it has been shown that if the manifold from general relativity were to be an approximation to the causal set, the causal set elements would have to be distributed evenly, according to a Poisson process ([3]), in the manifold. This led to the possibility of analysing how would a causal set look like.

However, if the causal set is the fundamental structure, it feels necessary to find the process which is at the origin of the causal set, and which would explain the structure we observe - a causal set dynamics. In recent years, a class of random models has been developed, the classical sequential growth models [4]). In these models, the causal set is grown one element at a time, with each new point being born in the future of some of the older elements. The process is similar to another random model: transitive percolation, or in a mathematician's vocabulary, random graph orders. In the following, we will investigate some properties of these models, relying on the better studied random graph orders. Finally, we will see that they can be considered as a part of a larger set of models assigning probabilities to causal sets, and several connected competing models are presented.

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Chapter 1

The formalism of Causal sets

1.1 Partial orders

Causal sets, or causets, are built on the mathematics of partial ordered sets. We will introduce here these mathematical objects, with a presentation inspired by [5].

1.1.1 Definition

An *order* is a quite common object in mathematics. For example, we use very often the order on integers and real numbers. These particular orders are called total orders, as it is possible to compare any two elements. This is not generally the case for a partial order which we define using binary relations.

Definition 1. A *binary relation* \mathcal{R} on a set X is a subset of $X \times X$. We write $x\mathcal{R}y$ iff $(x, y) \in \mathcal{R}$.

Definition 2. A *non-strict partial order* is a binary relation \mathcal{R} on a set X satisfying the conditions:

- $\forall x \in X, (x, x) \in \mathcal{R}$ (*reflexibility*)
- $\forall x, y \in X, ((x, y) \in \mathcal{R} \cap (y, x) \in \mathcal{R}) \Rightarrow x = y$ (*antisymmetry*)
- $\forall x, y, z \in X, ((x, y) \in \mathcal{R} \cap (y, z) \in \mathcal{R}) \Rightarrow (x, z) \in \mathcal{R}$ (*transitivity*)

We write $x \preceq y$ iff $(x, y) \in \mathcal{R}$.

Definition 3. A *strict partial order* is a binary relation \mathcal{S} on a set X satisfying the conditions :

- $\forall x \in X, (x, x) \notin \mathcal{S}$ (*irreflexibility*)
- $\forall x, y \in X, (x, y) \in \mathcal{S} \Rightarrow (y, x) \notin \mathcal{S}$
- $\forall x, y, z \in X, ((x, y) \in \mathcal{S} \cap (y, z) \in \mathcal{S}) \Rightarrow (x, z) \in \mathcal{S}$ (*transitivity*)

We write $x \prec y$ iff $(x, y) \in \mathcal{S}$.

We have given two definitions of partial orders, but they are essentially the same: actually we have $x \prec y \Leftrightarrow (x \preceq y \cap x \neq y)$. They are analogous to the well-known orders on integers or real numbers: \leq and $<$. By default, we will consider strict partial orders, more adapted to causal sets.

1.1.2 Examples

Example 1. In set theory, the inclusion defines a non-strict partial order. We can easily verify the three conditions. The partial order defined on $\mathcal{P}(\{0, \dots, n-1\})$ is called \mathcal{B}_n . In \mathcal{B}_2 , the order has 4 relations : $\emptyset \subset \{0\} \subset \{0, 1\}, \emptyset \subset \{1\} \subset \{0, 1\}$.

Example 2. In arithmetic, the relation of divisibility is a non-strict partial order. It is clearly reflexive (an integer k divides itself), verifies the antisymmetry condition (if a divides b and b divides a , $a = b$). If a divides b , $b = ak$, and if b divides c , $c = bl$. Hence $c = ak l$ and a divides c which proves transitivity.

Example 3. In \mathbb{R}^n , we can define a strict partial order as follows.

For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$,

$$x \prec y \Leftrightarrow \forall 1 \leq i \leq n, x_i < y_i$$

Hence we can compare two points if and only if the comparison of all coordinates gives the same result.

1.1.3 Additional definitions

We introduced partial orders using usual orders on integers and real numbers which are, as we said, total orders. In fact, a total order is actually a partial order, but it verifies an additional condition that all elements must be comparable.

Definition 4. Take a partial order $R \subset X \times X$. An **extension** R' of R is partial order such that $R \subseteq R'$. A **linear extension** is an extension which is a total order.

For a finite set, one can find an element which precedes all others (the minimal element), then the one which precedes all except the minimal etc. Hence, we can put them in that order on a straight line. This might help to see why they are also called *linear orders*.

Definition 5. A **minimal/maximal element** is an element which is preceded / precedes no elements (except itself for a non-strict order).

Points which can be put on a straight line form chains. Points that can't be in the same chain are in an antichain.

Definition 6. A **chain** is a subset $A \subseteq X$ such that the restriction of R on the subset R_A is a linear order.

Definition 7. An *antichain* is a subset $A \subseteq X$ such that $R_A = \emptyset$. A *maximal antichain* is an antichain such that there is no subset B such that $A \subset B \subseteq X$ and B is an antichain.

Definition 8. The *height* of a partial order is the maximal size of a chain. The *width* of the partial order is the maximal size of an antichain.

Some points are special. Alon et al. introduced the notion of a post ([6]) as we define in the following. It is remarkable as it is by itself an antichain and every maximal chain contains the post.

Definition 9. A *post* is an element which is related to every other element. Hence every other point falls either in its past or its future: it divides the set in two well defined parts.

Usual topologies can be complicated using causal sets. We will rely on the topology of intervals, which correspond to Alexandrov neighbourhoods i.e. volumes $J^+(x) \cap J^-(y)$ where J^\pm are causal future/past of a point.

Definition 10. The *interval* between x and y is the set $[x, y] = \{z | x \preceq z \preceq y\}$. The order is the induced order on the interval.

1.1.4 Representations

For finite sets, a binary relation can be represented in many ways. For partial orders, we have even more possibilities.

Representations of a binary relation

Let R be a binary relation on X , and n the cardinality of the set X .

We can define a $n \times n$ matrix M by

$$M_{ij} = \mathbb{1}_{(i,j) \in R}$$

This will be commonly used in programming. More visually, we can represent the binary relation as a directed graph. A directed graph is a pair (X, E) where X is the set of nodes and $E \subset X^2$ is the set of edges. Let G be the graph where the set of nodes is X and the set of oriented edges is R . This means that we draw a node for each element of X and we draw an arrow from x to y iff $(x, y) \in R$. What we get looks like Figure 1.1a.

Representations of a partial order

A partial order is by definition a binary relation. However, drawing all relations of a partial order can lead to overcharged drawings. To cope with this, we define *links*.

A link is a couple of related points $x \prec y$ such that there is no z verifying $x \prec z \prec y$. In other words, the interval $[x, y]$ is reduced to the set $\{x, y\}$. For any couple of related points x, y , one can find a set of points s_1, \dots, s_k such that $s_1 = x$, $s_k = y$ and $\forall i < k, (s_i, s_{i+1})$ is

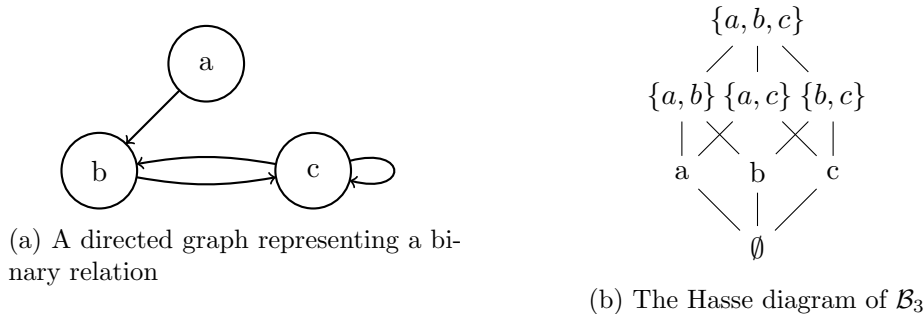


Figure 1.1: Representations of binary relations

a link. Hence if we know the links, we can compute the relations (this process is called "transitive closure") and if we know the relations, we can find among them the links. The knowledge of links is then equivalent to the knowledge of relations.

The proper definition uses the product of binary relations: given two relations R, S on X , we define $T = R \cdot S$ by

$$xTz \Leftrightarrow (\exists y \in X : xRy \cap ySz)$$

Then we can define $R^n = R^{n-1} \cdot R$ and finally, the transitive closure of a binary relation R is $TC(R) = \bigcup_k R^k$. One can show that the set of links L of a partial order P is a binary relation which verifies $TC(L) = P$. Actually, equivalently L is the smallest set (in terms of inclusion) which verifies that property.

Hence we only need to draw the links of the partial order on a directed graph to represent it. To make it a little more simple, we can draw the nodes in a way that for any related pair of points, the preceding is positioned lower than the succeeding. Hence, we don't have to draw the arrows on the links, as they become implicit with this convention. This drawing is called a Hasse diagram and an example is given in Figure 1.1b.

1.1.5 Graphs

We have already pointed out that a partial order can be represented using a directed graph. Actually, there is a one-to-one correspondence between directed graphs and binary relations. The image of partial orders is the set of transitive acyclic directed graphs. This bijection shows that the two objects are strictly equivalent and one can work with either interchangeably. This leads us to consider a few interesting graphs.

Definition 11. For a partial order (X, R) , the **directed graph of links** is $G = (X, E)$ with $E = \{(x, y) | [x, y] = \{x, y\}\}$.

Then, we can define the undirected graph of links. An undirected graph, or imply graph, is a pair (X, E) with X the set of nodes and $E \subset \mathcal{P}(X), \forall s \in E, |s| = 2$, the set of edges.

Definition 12. The **undirected graph** $G' = (X, E')$ is defined by $E' = \{x, y | (x, y) \in E\}$.

A given graph of links corresponds to many partial orders. However, if we fix a labelling, only one works. We construct the directed graph of links by

$$E = \{(x_i, x_j) | \{x_i, x_j\} \in E', i < j\}.$$

The partial order is the transitive closure of it.

1.2 Causal sets

Causal sets describe spacetime by a locally finite partial order. Locally finite means that $\forall x, y | [x, y] | < \infty$. The partial order represents the causal order in the spacetime; elements related by the order are "atoms of spacetime", the fabric of spacetime.

1.2.1 Causal sets from continuous spacetime

Causality in a Lorentzian manifold

In general relativity, we describe spacetime as a smooth manifold with a Lorentzian metric. We can draw curves on the manifold, which can be locally described as spacelike, timelike or lightlike. We say that two spacetime points are causally related if there exists a curve joining the two points which is everywhere either timelike or lightlike. Physically, an event in a point x unrelated to a point y can't influence nor be influenced by the event y .

One can easily see that this causal relation is a partial order, defined on the infinite set of spacetime. It has been shown that this structure can be considered as almost fundamental. In [2], Hawking et al. show that causal structure define the metric of spacetime up to a conformal factor.

The assumption of causal sets is that spacetime isn't continuous, but discrete and locally finite. In every volume there is a finite number of spacetime elements. These elements are related with the causality relation described above, hence forming a partial order which encodes the causal structure of spacetime. Finally, the conformal factor is replaced by the number of points: every point is a unit volume, so counting points in an interval gives the volume of the interval. The set of spacetime elements along with the partial order is the causal set and is expected to be a fundamental discrete description of spacetime.

Sprinkling

Direct experience has not yet proved the existence of the assumed spacetime points, hence we do not know exactly how are they distributed in space. However, for Minkowski spacetime, an additional assumption is that Lorentz invariance has to be preserved. In that case the only solution ([3]) is that the points must be randomly distributed, according to a Poisson distribution. The process of taking random points in this way is called sprinkling and gives an idea of what a causal set has to look like.

One of the basic efforts for causal sets is to find a dynamic which defines a causal set that is similar to the result of a sprinkling.

1.2.2 Continuous spacetime from causal sets

As we have said, the assumption is that the fundamental object is not the spacetime but the causal set, which is discrete and locally finite. More exactly, spacetime doesn't exist as we know it; the causal set is spacetime. This comes as kind of contradictory to our everyday experience of continuity. Causal sets have an answer to that problem.

The idea is that in the same way as we might take a discrete approximation to a continuous space, a continuous space can be an approximation to a discrete set. We can have many discrete approximations to a continuous object; equivalently we can have many continuous approximations to a discrete set. What we have to assume is that any two approximations to a causal set are "similar", almost surely or with very high probability, possibly assuming they respect some conditions. This assumption is referred to as *Hauptvermutung*. There is currently no known proof of this conjecture, which is commonly accepted. An analogy would be the existence of a continuous approximation for a fluid which is in reality a set of molecules.

Hence we do not observe discrete structure in everyday life because the discreteness exists at such a small scale. On our scale, things happen as if the spacetime was a continuous manifold. But although we can't experience it directly, we can test if a manifold is a good approximation for a causal set. As spacetime inherits causal structure from the causal set, we expect that a "good approximation" is such that a sprinkling might give back the initial causal set. Hence, the causal set needs to have a structure which can be mapped on the manifold. This map is called an embedding.

Definition 13. An *embedding* of a causal set (C, \prec) in a Lorentzian manifold M is a map

$$f : C \rightarrow M$$

such that

$$\forall x, y \in C, x \prec y \Rightarrow \text{there is a causal curve from } f(x) \text{ to } f(y)$$

It is a causal isomorphism (see Definition 17) from (C, \prec) to (M, \prec_M) where \prec_M is the causal structure of M derived from its Lorentzian metric.

A *faithful* embedding is one which has a constant density of in the manifold, consistent with a Poisson distribution. An embedding of a causal set is a first step; if we want to get the conformal factor right, we need to have a faithful embedding.

Chapter 2

Sequential Growth Models

As we said, one of the challenges of causal sets is to find a way of constructing a causal set without reference to any underlying spacetime: a dynamics for causal sets. This should in principle be describing spacetime and ideally quantum mechanics, as well as many features of the universe we experience. The difficulty of this is mind-boggling; so a first step is to try some dynamics and see what features can be extracted.

One can imagine many possible ways of building a causal set. Among them, sequential growth models are one radically different than sprinkling. We will now build a causal set by iteration, adding points step by step and adding relations while conserving what was built before.

In the following, we will create one point at a time, add relations to the point and calculate the transitive closure to get a partial order at each step. We will use usually a causal set C of cardinality N , with points numbered $0 \dots N - 1$, called by their number k or by x_k , and the birth of a new element number N . We will select a set of ancestors; it is included in the past of N , which is then completed by transitive closure. The newly formed causal set will then be C' of cardinality $N + 1$. At every step, existing elements and relations are conserved, which explains why we call them "growth models".

2.1 Definition

In [4] Sorkin and Rideout construct a class of stochastic Sequential Growth Models which stand on 3 "reasonable" conditions.

- Internal temporality: any new element can't be born to the past of an existing element. Hence an older element can only precede a younger one (older and younger as in the growth process). If one assigns numbers to elements of the partial order in the order of birth, one gets a linear extension of the partial order.
- Discrete general covariance: the order in which elements doesn't matter on the probability of the resulting causal set. The order of birth, being an "external time" isn't then a relevant coordinate in the causal set.

- Bell causality: what happens at a point can only be influenced by what happened in its past.

Also assuming that the considered transition probabilities do not vanish, they derive a class of models which we will discuss now. A sequential growth model corresponds to a sequence of numbers (t_n) which defines the probabilities of transition from any N element causet to any $N + 1$ element causet which is obtainable by growth. The process is the following:

- At step N , we have an N -element causal set C and we create a new element
- Select an "ancestor set" $A \subset C$ with probability proportional to t_k if A is of cardinality k
- the new born is placed in the future of A , and of all points in the past of A - the past of the element is now fixed for the rest of the growth and is called the predecessor set

Let's stress that we don't have an ancestor set of size k with probability $\propto t_k$, but that every set of size k is chosen with probability $\propto t_k$. For example, the new element has no ancestors with probability proportional to t_0 or it's ancestor set is a fixed 3 element set with probability proportional to t_3 etc.

2.1.1 Probabilities implied by the definition

There are $\binom{N}{k}$ subsets of cardinality k , which leads us to a normalization by

$$\sum_{i=0}^N \binom{N}{i} t_i.$$

Hence the exact probability to have a given ancestor set of cardinality k at step N is

$$\frac{t_k}{\sum_{i=0}^N \binom{N}{i} t_i}$$

We have assigned a probability to every possible ancestor set. But as any element can be in many ancestor sets, the probability of having a specific element in the ancestor is calculated differently. Fix an element a in C . a can be the only ancestor, with a probability proportional to t_1 ; it can be one of the two ancestors in any of the $(n - 1)$ two-element subsets containing a ; it can be in one of the $\binom{N-1}{k-1}$ k -element ancestor sets containing a . Hence, the probability of having a in the ancestor set is the sum of these, namely:

$$\frac{\sum_{j=1}^n \binom{n-1}{j-1} t_j}{\sum_{i=0}^n \binom{n}{i} t_i} = \frac{1}{n} \frac{\sum_{j=0}^n \binom{n}{j} j t_j}{\sum_{i=0}^n \binom{n}{i} t_i} \quad (2.1)$$

Given a set of ancestors $a_1 \dots a_l$, the probability of having them among the ancestor set is

$$\frac{(n-l)! \sum_{j=0}^n \binom{j}{n} \frac{j!}{(j-l)!} t^j}{n! \sum_{i=0}^n \binom{i}{n} t^i} \quad (2.2)$$

Now the probability of a point to be in the predecessor set is much more complicated as it depends on the structure of the set. For example, a single point disconnected from the set can only be a predecessor if it is in the ancestor set; on the opposite, if the set has a single minimal point, it will be a predecessor as long as the new point has any ancestor. However, there is a general rule. In the growth process, the future of any point is always increasing; and if the new born has an ancestor in the future of a point, by transitivity they are all related. So in a relatively general way, the older the point, the more likely it precedes the new point. This intuition shouldn't be taken as more than it is: it depends on the point and on the t_n sequence. But we can see for example, that if we have a post in the causal set, it will stay a post as long as every new element has at least one ancestor in the future of the post. Our intuition shows that this probability increases, so the longer a point stays a post, the more likely it will continue being one.

2.1.2 Common factor invariance

The probabilities that we use are only proportional to the sequence t_n . Actually, we might have directly given the probabilities of the transition, just as we calculated it afterwards. However we would have lost in generality, as the probabilities change with the normalization, at every step.

The price to pay is that the probabilities have only a meaning in a relative way. Hence, there is a free parameter for every model: a common factor. The sequences (t_n) and (αt_n) give exactly the same results. We can then fix any value (as they are all non vanishing) to 1 - we fix usually $t_0 = 1$.

2.2 Transitive percolation

Transitive percolation is a particular growth model where $t_n = t^n$ with a fixed parameter t . It has a lot of interesting features.

2.2.1 Point independence

Let us calculate the probability of a fixed point to be in the ancestor set of N . Using equation 2.1, we get

$$\frac{1}{n} \frac{\sum_{j=0}^n \binom{j}{n} j t^j}{\sum_{i=0}^n \binom{i}{n} t^i} = \frac{1}{n} \frac{((1+t)^n)'}{(1+t)^n} = \frac{t}{1+t} \quad (2.3)$$

This probability does not depend on n : let's define $p = \frac{t}{1+t}$. What we get is that at every step, the new element chooses his ancestors each one independently with probability p . Actually, this is why we call it transitive percolation : in random graph order theory,

percolation consists in taking n nodes and adding to the set of edges any possible edge with probability p . What we have just shown is that this can also be done with partial orders, in the formalism of sequential growth models. Now we can either see transitive percolation as sequential growth model, or we can see sequential growth models as "transitive percolation" where edges are added to the set of edges with a varying probability. Equation 2.1 shows that at each new element, the probabilities of every edge are equal - the probability of an edge $i \prec j$ depends only of j .

We set up a set of n elements and assign them numbers $0..n-1$. We take all pairs of elements and, with probability p , add a relation $x_i \prec x_j$ when $i < j$. This gives exactly the same as transitive percolation stopped after n steps.

2.2.2 Order reversal symmetry

Transitive percolation is symmetric under order reversal in the following sense.

Definition 14. *Order reversal is the function I which associates to a causal set $C = (X, \prec)$ the causal set $C' = (X, \succ)$.*

$$I : C = (X, \prec) \longrightarrow C' = (X, \succ)$$

Which means that if $x \prec y$ in C , then $y \prec x$ in $C' = I(C)$.

Definition 15. *A model is order reversal symmetric if it gives with the same probability a causal set C as its order reversed $I(C)$.*

We will see in 5.1.2 that the probability of a given causal set depends only on the number of acceptable labellings, number of links and number of relations. The number of links and the number of relations is obviously the same after inversion. The number of acceptable labellings is also the same, as we just need to reverse labels ($i \rightarrow n-1-i$). Hence transitive percolation is order reversal symmetric.

Moreover, transitive percolation is the only sequential growth model which is order reversal symmetric. To show this, consider the set defined with a single minimal element with n points in its future, all mutually unrelated. The probability of getting this is, up to the normalisation factor which is the same for all $n+1$ point causal sets, equal to

$$t_0 \cdot t_1 \cdot t_1 \cdots t_1 = t_0 t_1^n$$

As the first point has no ancestors and every other point has a unique ancestor (see Figure 2.1a).

Now, the symmetric of this set is the one with n minimal points, all in the past of a single maximal point. The probability of getting this is, up to the normalisation factor,

$$t_0 \cdot t_0 \cdots t_0 \cdot t_n = t_0^n t_n$$



Figure 2.1: Illustrations for Section 2.2.2

As we first need to build all minimal points to create one in the future of all other (see Figure 2.1b).

Now we know that the coupling constants are defined up to a constant factor. We can then assume that $t_0 = 1$. Let's call then $t_1 = t$. The assumption we made at the beginning states then that

$$\begin{aligned} t_0 t_1^n &= t_0^n t_n \\ t^n &= t_n \end{aligned} \tag{2.4}$$

This means that the growth model is transitive percolation.

2.3 The partial order of causal sets

As we have explained, causal sets are partial orders. But the set of causal sets can also be thought of as a partial order. We define the relation \mathcal{R} as following:

$$C\mathcal{R}C' \Leftrightarrow \text{one can grow } C' \text{ out of } C \text{ in a sequential growth process}$$

This definition doesn't depend on the particular model since we have assumed that no transition probability vanishes. One can easily see that this corresponds to a partial order: for example if we can grow B from A and C from B , we can grow C from A , thus proving transitivity.

The set of causal sets with this relation is a partial order, and is commonly referred to as "poscau". Sequential growth models assign probabilities to the links of poscau, as links correspond to additions of single points. They can be thought of as a part of a larger group of models which assign these probabilities to the links. Sequential growth models form the part which respects the conditions given in [4].

Generally, a causal set can be obtained through several different sequences of transitions. As any set of transitions corresponds to a walk on poscau, we say that these sequences of transitions are "paths". This leads to the addition of a feature to poscau. Sometimes, several different transitions can lead to the same resulting causal set. By construction, they are the same link on poscau, but they are different paths - an example is given in Figure 2.3. Hence we label links with the number of different transitions that it corresponds to (the default 1 is usually not written).

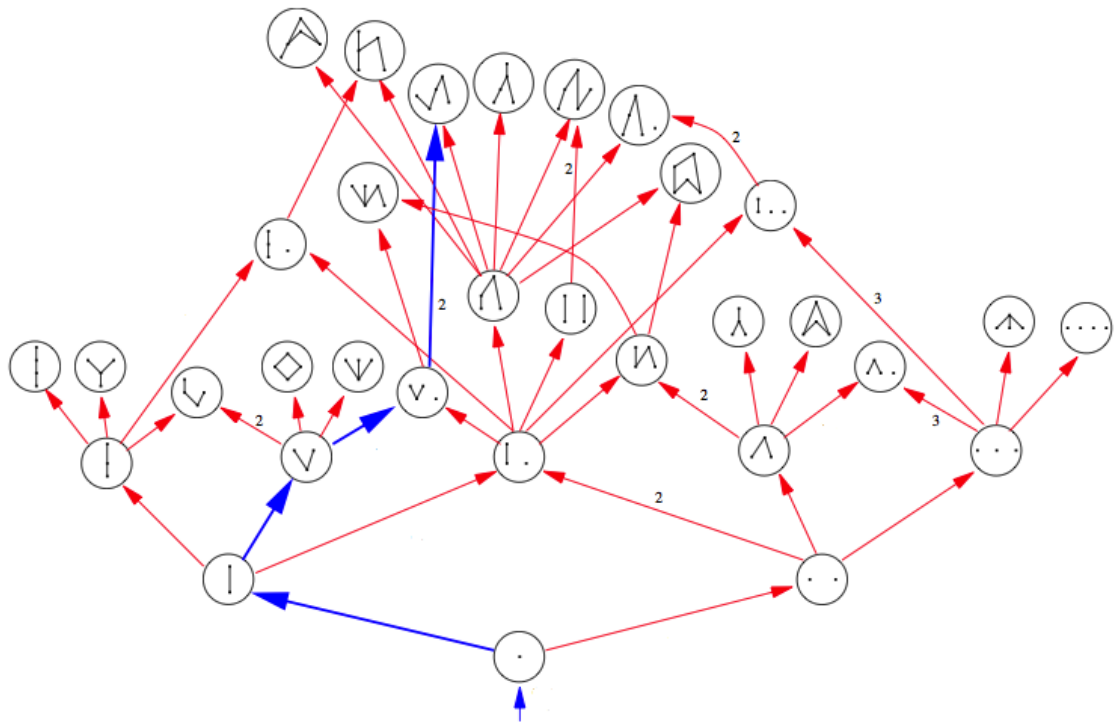
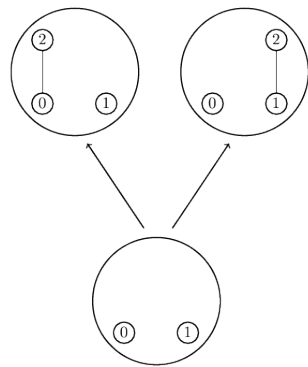
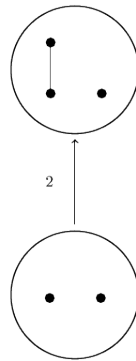


Figure 2.2: Partial order of finite causal sets, with a specified path in blue. Obtained from [4, Figure 1].



(a) Two possible paths



(b) One link

Figure 2.3: A causal set evolving through two paths to a single causal set

Chapter 3

Properties of transitive percolation

There are many reasons to work on transitive percolation in particular among other sequential growth models. Firstly, it is one of the simplest models we can choose, both conceptually (simple rules, uses independent random variables) and from the point of view of computing (easy implementation). Secondly, it has many interesting features, as we will see in this section. Additionally, we have many theoretical results, since transitive percolation has been much more extensively studied than any other sequential growth model. The results come from work on random graph orders by many mathematicians, most importantly Bollobas ([7, 8, 9, 10]) and Brightwell ([7, 8, 9, 10, 11, 12]).

In random graph orders, one considers partial orders of size n resulting from percolation with parameter $p = p(n)$. As percolation is a random process, the partial order is itself a random variable. Many results exist on the limit $n \rightarrow \infty$. Usually, p is changing with n but not in the same way as in sequential growth models:

- In random graph orders, for every size n we do a percolation with parameter $p(n)$
- In sequential growth, at every step k of a n -step process, probabilities of transition depend on k

By considering $p(n) = \text{constant}$, we get results on transitive percolation as we see it in sequential growth models. We can also get results for "small p " considering $p \rightarrow 0$.

3.1 Width of a grown causet

The width of a partial order is an important characteristic of a partial order from a physical point of view. Indeed, the width of a causal set corresponds to the size of its largest antichain, and since antichains are physically spacelike surfaces, the width of the causal set gives the size of a largest spacelike slice of the universe.

In [7], the width of partial orders is given for certain values of p and n . Two main results are given:

- If $p \geq \frac{(50 \log \log n)^2}{\log n}$ then the width of the partial order is almost exactly $\sqrt{\frac{2 \log(n)}{\log(1/q)}}$
- if $p \log n \rightarrow 0, pn \rightarrow \infty$ then the width is almost surely close to some function of the form $C(p)p^{-1}$, with $1.455 < C(p) < 2.428$, where $C(p)$ depends on n only via p .

The first assertion applies to transitive percolation if we fix p to a value and take $n \rightarrow \infty$. Then we will have for some n_0 that if $n \geq n_0$ then $p \geq \frac{(50 \log \log n)^2}{\log n}$. Now using

$$\log(1/q) = \log\left(\frac{1}{1-p}\right) = \log(1+p+o(p)) = p+o(p)$$

we get that the width of the partial order is slowly growing in average, with a value close to $\sqrt{2 \log(n)p^{-1}}$.

The second applies when we consider $p \ll 1$. Then, for relatively large values of n , we have still $p \log(n) \ll 1$ while $pn \gg 1$ and the second assertion is approximated. Hence, the width is of order p^{-1} .

The first result gives a width of order $p^{-1/2}$, while the second p^{-1} . This is not contradictory. While the second applies to a large number of causal sets, with various values of p and n , the first applies only to sets with very large n . We should see this the following way:

- if we take a small p and a large n , the width will be of order p^{-1} ;
- if we fix a p and grow the causal set, the longer we wait the higher values of width we get.

In a way, if we divided a long set in parts, each part would have a width of order p^{-1} , but the maximal value among them is of order $\sqrt{2 \log(n)p^{-1}}$.

We will develop here a different perspective. The width, as it has been defined, is a global characteristic of the causal set and hence it isn't evolving with the causal set growth. It is defined at every step, but there is no relation between maximal antichains at two following steps. Indeed, it might happen that for a long "time" the maximal antichain is fixed, and then changes abruptly to a completely different place.

Our point of view here will be of someone "travelling" along the new born elements (although this has no physical meaning in causal set theory - the "path" is not timelike). The new born element has no future yet, so it is a maximal element. Hence it is part of an unextendible antichain: the set of maximal elements (this can be easily verified). We will look at the size of that specific antichain. Now this antichain looks like if it was evolving "in time". At every step, it can grow by only one point, the new born, and it loses only the points that fall in the past of the new born. In a way, the new born replaces some of the maximal elements.

Let us call M_n the number of maximal points in the causet after n steps (the causal set has n elements). After the step $n+1$, we get one new maximal element, and we lose

all those which fell in the past of the new one. Let's call X_n this number: we get

$$M_{n+1} = M_n + 1 - X_n.$$

In transitive percolation, every relation is added independently with probability p . If we fix some k out of m maximal elements, the probability of the new born to fall in their future is $p^k q^{m-k}$. We finally get that:

$$\mathbb{P}(X_n = k | M_n = m) = \binom{m}{k} p^k q^{m-k}.$$

The expected value of that is pm . Hence, if $M_n \rightarrow m$, we get

$$m = m + 1 - pm.$$

Which gives us the relation $m = 1/p$.

This shows that the number of maximal points, which starts at 0, will grow approximately to $1/p$ and then fluctuate around that value. We get then a result of the same order as for the width of the partial order. We get this way some new insight on the evolution of the width with n . As the number of maximal points is a random variable fluctuating around $1/p$, its maximum will be growing with n (it can't decrease) but very slowly (the probability of getting large values decreases rapidly).

Physically, this shows that the number of maximal points gives a good approximation to the spacelike size of the universe. Hence, we observe that the size of the universe is usually of order $1/p$. The problem is that if we wished transitive percolation to be an exact model for our universe, the size of visible universe implies a greater value for p : the 10^{240} spacetime points need $p \leq 10^{-240}$.

3.2 Dimension of a grown causet

As we explained in 1.2.2, the idea behind causal sets is that continuous spacetime is an approximation to the fundamental causal set. If we had a manifold realising this approximation, then we could embed the causal set in the manifold. The manifold having a dimension, the causal set would also inherit from that dimension. And as we expect that all approximations are, in a way, similar, then we expect that their dimension would be basically constant. Hence, the dimension could be defined without reference to any manifold and would be then a characteristic of the causal set. Also, if we knew how to define the dimension of the causal set, we would know among which manifolds should we look for a good approximation. This motivates the following section, largely based on [3].

3.2.1 Minkowski dimension

The reason we look for a causal set dimension leads to a first definition. As any Lorentzian manifold is locally flat, we first want to learn about embedding causal sets in Minkowski space.

Definition 16. *The **Minkowski dimension** of a causal set is the smallest dimension of a Minkowski space in which we can embed the causal set.*

This definition refers only to the causal structure, but not to the density (faithfulness) of the embedding.

3.2.2 Hausdorff dimension

Although the definition is simple, it isn't easy to use. For instance, one can't test every manifold to find the smallest dimension, which makes it difficult to compute. That's why another dimension is used for causal sets: its Hausdorff dimension. Hausdorff dimensions are used commonly to evaluate a dimension-like characteristic of an object which has a complicated, non-continuous geometry, such as fractals.

The basic idea here is that the volume of a causal interval in Minkowski space depends on the proper time between the two events defining the interval, and on the dimension of the space.

$$V = \frac{V_{d-1}}{d+1} \left(\frac{T}{2}\right)^{d+1}$$

where T is the proper time, d is the dimension and V_{d-1} is the volume bounded by a unit S_{d-1} . This relation can be inverted if we take d to be a real parameter, hence we can define d as a function of V and T . This leads to estimators of dimension which do not refer to any geometry - as long as V and T can be defined, we can find d . Interestingly, they can be defined for a causal set: the proper time between two elements corresponds to the length of the longest chain between them, and the volume of their causal interval corresponds to the number of elements in the causal interval of the causal set.

In [3], Meyer develops an estimation of the dimension using the preceding observation. His estimator, close to results by Myrheim, relates the expected number of 2-chains in an interval, the volume of the interval and the dimension for a causal set faithfully embeddable in Minkowski spacetime (2-chains are chains of length 2, starting and ending at the points which define the interval)

$$\langle C_2 \rangle = \langle V \rangle^2 \cdot f(d)$$

Where C_2 is the number of 2-chains, $\langle V \rangle$ the volume of the interval and $f(d) = \frac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)}$. We create the estimator when we take specific values of C_2 and V and find the value of d which realises the equality. As Meyer shows in [3, Figure III.4], f is invertible,

hence if we define the inverse f^{-1} we get the Myrheim-Meyer estimator

$$d \equiv f^{-1} \left(\frac{C_2}{V^2} \right)$$

In [13], we have a reference to another dimension estimator relying on a similar principle. Take an interval $[x, y]$ of size N and take a point $z \in [x, y]$. Consider N_1, N_2 the sizes of $[x, z]$ and $[z, y]$ and define $N_s(z) = \min(N_1(z), N_2(z))$. Finally define $N_{mid} = \max_z N_s(z)$. One can easily see that in Minkowski space we would have the corresponding z in the middle between x and y . If the proper time between x and y is τ , the volume of $[x, y]$ is $\tau^d \propto N$; the proper time between x and z is $\tau/2$ so the volume of $[x, z]$ is $(\frac{\tau}{2})^d \propto N_{mid}$. Hence we define the midpoint scaling estimator:

$$d \equiv \log_2 \left(\frac{N}{N_{mid}} \right)$$

3.2.3 Combinatorial dimension

These estimators give only an approximation of the dimension, since the results depend on the points one considers. Also, they give real values which only suggest integer values for an approximate manifold. That's why we may be tempted to use another dimension: the combinatorial dimension.

For 2d Minkowski space, with coordinates x and t , we can define coordinates $u = t - x$ and $v = t + x$. The causal structure in these coordinates is the following

$$x \prec y \Leftrightarrow (u(x) < u(y) \cap v(x) < v(y)).$$

In the language of partial orders, this means that the partial order is the intersection of two total orders: the total order associated to u and the one associated to v . By definition, the intersection of total orders is the partial order which has exactly the relations which are common to the total orders. Then we define the combinatorial dimension as the smallest cardinality of a collection of total orders whose intersection is the partial order. We have just seen that a causal set embeddable in Minkowski space has combinatorial dimension less or equal 2. The converse holds too. Hence causal sets of combinatorial dimension 2 and Minkowski dimension 2 are the same. Unfortunately, the relation doesn't hold for larger dimensions; geometrically, Minkowski light cones are cones, while the intersection of total orders gives rise to "pyramidal" light cones. However, there is speculation on the relation between the two: for example, in [3], there is a conjecture that the Minkowski dimension is less or equal to the combinatorial dimension. Today, we know this not to be true. Results from [14] show the existence of 3 dimensional partial orders (in a combinatorial way) which are not circle orders. This restricts the relations one can get between Minkowski and combinatorial dimensions, but studying combinatorial dimension can still be fruitful.

The combinatorial dimension has been studied much more than Minkowski dimension and thus we have many interesting results. For instance, we know that the combinatorial

dimension is always finite (it is smaller than the number of elements. From the point of view of transitive percolation, we have important results from work on random graph orders. Most importantly, we have in [9] that almost surely:

$$(1 + \epsilon) \sqrt{\frac{\log(n)}{\log(1/q)}} \leq \dim P_{n,p} \leq (1 + \epsilon) \sqrt{\frac{4 \log(n)}{3 \log(1/q)}}$$

Since $1/q = 1/(1-p) = 1 + p + o(p)$, for small p we have $\log(1/q) = p + o(p)$ and hence an approximate lower value for the combinatorial dimension of transitive percolation with parameter p and n steps is $\sqrt{\frac{\log(n)}{p}}$. This is growing with n , but very slowly. More importantly, it is proportional to $p^{-1/2}$, hence for small values (which lead to large universes), the dimension might become very high.

3.3 Posts

3.3.1 In transitive percolation

When we defined posts (see Section 1.1.3), we have made explicit the importance of such points in the structure of a partial order. Such a point divides the set into its past and its future. This has an important meaning from a physical point of view. In [15], Sorkin interprets them as the point where a "Big Crush" becomes a "Big Bang" in a "bouncing" cosmology. Indeed, such a point marks the change between the old universe that collapsed and the new era which expands. Furthermore, from the point of view of someone in the future of a post, it seems that everything came out of a single spacetime point.

Counting posts

In transitive percolation, there is an infinite number of posts in an infinite causal set. To see this, let's evaluate the probability of having a post at step N . The probability that N is a post is equal to the product of the probability that when it is born, it falls in the future of all existing points and of the one that everything which follows will be born in its future.

To evaluate the first one, we can just remember that the number of maximal points is expected to be of order $1/p$. Since falling in the future of all existing points is equivalent to having all maximal points in the ancestor set (other relations will be added by transitive closure), the probability that N is in the future of all $j < N$ is of order:

$$p^{1/p}.$$

The probability of N having all $j > N$ in its future is as follows. The probability that j is in the future of N given that all $i : N < i < j$ are in the future of N is $1 - q^j$. So the

probability that all $i : N < i \leq j$ are in the future of N is

$$\prod_{i=1}^j (1 - q^i) \longrightarrow \prod_{i=1}^{\infty} (1 - q^i) \equiv \eta(p)$$

We can evaluate simply $\eta(p)$

$$\begin{aligned} \log \eta(p) &= \sum_{i \geq 1} \log(1 - q^i) = \sum_{i \geq 1} \sum_{k \geq 1} (-1)^{k+1} \frac{(-q^i)^k}{k} \\ &= - \sum_{k \geq 1} \frac{1}{k} \sum_{i \geq 1} q^{ki} \\ &= - \sum_{k \geq 1} \frac{1}{k} \frac{q^k}{1 - q^k} \end{aligned} \tag{3.1}$$

The sum we obtain is converging. We have thus proved that the probability of any element being a post is non vanishing. This implies that there is an infinite number of posts.

The same is proven in [8] but with a slightly different approach. A more general result is proven: we define $Y(n, p) = np^{-1}e^{-\pi^2/3p}$. If $Y(n, p) \rightarrow \infty$ then there is almost surely an infinite number of posts; if $Y(n, p) \rightarrow 0$ then there is almost surely no posts. We apply this to our transitive percolation where p is fixed and we can easily see that we are in the first case, which shows that there will be an infinite number of posts in the causal set.

Transitive percolation cosmology

In [15], Sorkin argues that it is an indication on what a causal set cosmology might look like. If it was built using a model similar to transitive percolation, like sequential growth models, we could expect an infinite sequence of eras, all starting with a Big Bang and ending with a Big Crush. This viewpoint is supported by the fact that in [16] it is shown that the initial evolution of ordinary transitive percolation (which is equivalent to transitive percolation after a post) has two phases.

- A tree-like phase, where the small value of p makes it unlikely for the new born to have more than one ancestor. The result is a spacetime where every interval is a chain, hence it is one-dimensional.
- An exponential growth phase where spacetime is evolving like deSitter space. This looks like an expanding universe we are living in.

In this "bouncing" cosmology, we can see then a few phases. After a post, tree-like growth is followed by deSitter type expansion up to a width of around $1/p$. Then, we have a period of fluctuating width, which ends up with a collapse - and since we have shown that transitive percolation is order reversal symmetric, we can assume that the collapse has also two phases, first exponential decrease then inverse-tree phase.

Interpost distance

In this part, we will see how big is the interval between two posts: the interpost distance. It measures the number of elements that fall between two posts. With this, we can better understand the structure of the fluctuation era.

We can approximate that using another result from [8]. The article shows:

$$d_{TV}(M_n, Po(\lambda)) \leq 21p$$

Where

- M_n is the number of posts
- d_{TV} is the total variation distance defined by $d_{TV}(X, Y) = \max_S |\mathbb{P}(X \in S) - \mathbb{P}(Y \in S)|$, where S are all subsets of the set of values taken by X and Y
- Po is the Poisson distribution
- $\lambda = n\eta(p)^2$

This means that for small p and large n , the number of posts follows a Poisson distribution of mean λ . Therefore, the frequency of posts is roughly λ/n , and equivalently the mean interpost external time is $n/\lambda = \eta(p)^{-2}$. Now the paper takes from other articles the approximation $\eta(p)^2 = (1 + o(1))2\pi e^{\pi^2/6} e^{-\pi^2/3p} p^{-1}$

Hence we get the interpost distance approximately equal to

$$f = \frac{e^{-\pi^2/6}}{2\pi} e^{\pi^2/3p} p$$

Apart from the constant, we have a term exponential in p^{-1} and a linear term p . Also, if we wanted to get the approximate physical length, in terms of time between two posts, we would need to divide by the mean width of the set, which is of order $1/p$. The constant and the two linear terms can be easily disregarded when considering small p - the exponential factor is much larger. Calculations give the interpost lengths shown in table 3.1. We can easily observe the exponential growth of the interpost length. This has a large impact on the cosmology of transitive percolation. For example, if we wished the maximal width to be of order the size of the visible universe (10^{240} in Planck units), we would need $p = 10^{-240}$, which would give an interpost time of roughly $10^{10^{240}}$, which is certainly a lot. However, as long as its finite, for an infinite universe, this gives infinitely many posts, hence infinitely many expansion periods.

Let's conclude with two remarks about that result. First, one shouldn't consider transitive percolation as the ultimate growth model for causal sets. Indeed, even at the definition level, sequential growth models do not include quantum interference. Second, as long as a model shows periods which look like our universe (the exponential growth) we can defend the model by saying that we got lucky to live in that part - although this isn't completely satisfying.

p	interpost length
0.1	$6.0 \cdot 10^{11}$
0.01	$2.3 \cdot 10^{139}$
0.001	$1.8 \cdot 10^{1424}$

Table 3.1: Mean interpost length

3.3.2 In sequential growth models

Counting posts in sequential growth models

In 2004 at the Workshop on Causal Sets in Dublin, Graham Brightwell presented the following result ([11]).

If $t_n = (\frac{\alpha}{\log(n)})^n$, then $\mathbb{P}(r \text{ is a post}) \sim C \log(r) e^{-\pi^2/3\alpha}$. Then

- there is almost surely an infinite number of posts if $\alpha > \pi^2/3$
- there is almost surely a finite number of posts if $\alpha < \pi^2/3$

Using the results on transitive percolation we reviewed in 3.3.1, we can find indications on why this is true.

Let's assume that $t_n = (\frac{\alpha}{\log(n)})^n$. Then we have $t_n = t^n$ with t slowly varying with n . We make here the approximation that everything happens as if t was constant. Then it corresponds approximately to a transitive percolation with parameter $p = \frac{t}{1+t}$. As $t \rightarrow 0$, we will assume t small and hence $p \simeq t$. Now the transitive percolation condition depends on $np^{-1}e^{-\pi^2/3p}$. When introducing back the dependence in n , we get:

$$\begin{aligned}
np^{-1}e^{-\frac{\pi^2}{3p}} &= n \frac{\log(n)}{\alpha} \exp\left(-\frac{\pi^2 \log(n)}{3\alpha}\right) \\
&= n \frac{\log(n)}{\alpha} n^{-\pi^2/3\alpha} \\
&= \frac{\log(n)}{\alpha} n^{1-\pi^2/3\alpha}
\end{aligned} \tag{3.2}$$

This goes to infinity if the power of n is positive, and to zero if it is negative. We have

$$1 - \pi^2/3\alpha > 0 \Leftrightarrow \alpha > \frac{\pi^2}{3}$$

Which is the result we expected, however with a reasoning which wasn't mathematically rigorous. The reasoning could probably be adapted to become rigorous. Indeed, it is easy to see that the longer a post stays a post, the more likely it will be one forever. Hence the probability of having a post depends mostly on what is happening "close" to the point, and for a slowly varying t we can expect it to be comparable to transitive percolation.

Cosmological renormalization

In [15, 17] an idea of renormalization of p after each post is developed. The idea is that if a post appears at the step N , then what follows may be approximated by transitive

percolation, with p depending on N . Then each era would have its own parameter p , and at each Big Crush the change of p could be seen as "renormalization" of the constant.

For example, in [17] Dou considers $t_n = \frac{t^n}{n!}$ and shows that after a post at step N , the evolution is similar to transitive percolation with a new parameter $\hat{t} = \frac{t}{\sqrt{N}}$ (and $p \simeq \hat{t}$). In [15] Sorkin suggests that this might be the explanation of a hierarchy problem, namely that for having a very large universe, we need a small parameter p (something we have seen in Section 3.1). Using the renormalization, we could imagine an initial parameter close to 1 being successively decreased at each post to arrive to the value needed to observe today's universe. This gives an elegant explanation on why do we see a universe with such fine-tuned parameters: it is because we are not living in the first era, but in one of an infinite sequence.

However, applying our result, we find a restriction on sequential growth models if we want them to have an infinite number of posts. For example, with $t_n = \frac{t^n}{n!}$ we will have only a finite number of posts, so it is wrong to assume that we can have a post at any value of N . The assumption used in the calculation, that N is large may not hold.

3.4 A global view of transitive percolation

Results from this section along with results from [8] help us to see what does a causal set from transitive percolation look like.

The causal set is a tube of varying width around p^{-1} . We can divide it in "slices" of size p^{-1} . A point is generally related to an unique point in the following slice; it is linked to one more in the next slice but the one already linked will also have one relation etc. Hence the number of points in each slice which are related to the first point is exponentially growing, and after $\log p^{-1}$ slices almost all points are related to the point. Hence we have a buffer of order $p^{-1} \log p^{-1}$ before and after any point where points are quite unlikely to be related to the given point, and all others are related. The dimension, globally looks to be 1. Very rarely we get a post, which cut the tube in very long parts (compared to the width). After a post, everything looks like transitive percolation conditioned to have a minimal point - ordinary percolation. This has two phases: a tree-like growth followed by exponential expansion up to the final width p^{-1} .

Chapter 4

Evolution of causal sets

In the preceding chapters, we've been interested in some properties of causal sets obtained from sequential growth models and in particular, transitive percolation. Here, we will apply simulation techniques to verify some of the results, and more importantly, we will look at the variations of the properties on the causal set. This way lets us distance ourselves from the mathematics' static point of view to a more physical evolution. In a way, calculating local values for width or dimension brings us closer to the physical experience of what would an observer see if he travelled along the causal set.

For this part, we have developed a *Java* project to do the simulations.

4.1 Evolution of the width

We defined the width in Section 1.1.3 as the size of the largest antichain. This is a global definition; we would prefer to define a "width at a point x ", for example as the size of the largest antichain containing x . However, for simulation ease, we will compute the number of maximal points when x is born. This is of course a rather bad approximation, firstly because the largest antichain may be different from the maximal elements antichain and secondly because future evolution can give a larger antichain. However, we have seen in Section 3.1 that the number of antichains is of same order as the width of the causal set, at least when n is large.

The result is in Figure 4.1. In a first phase, the growth is linear - just like in tree-like evolution, where the number of maximal points is of same order as the total number of points. Then the growth slows down, which should correspond to the exponential expansion. Finally, the growth is stabilised around $200 = 1/0.005$, and the width fluctuates around that value.

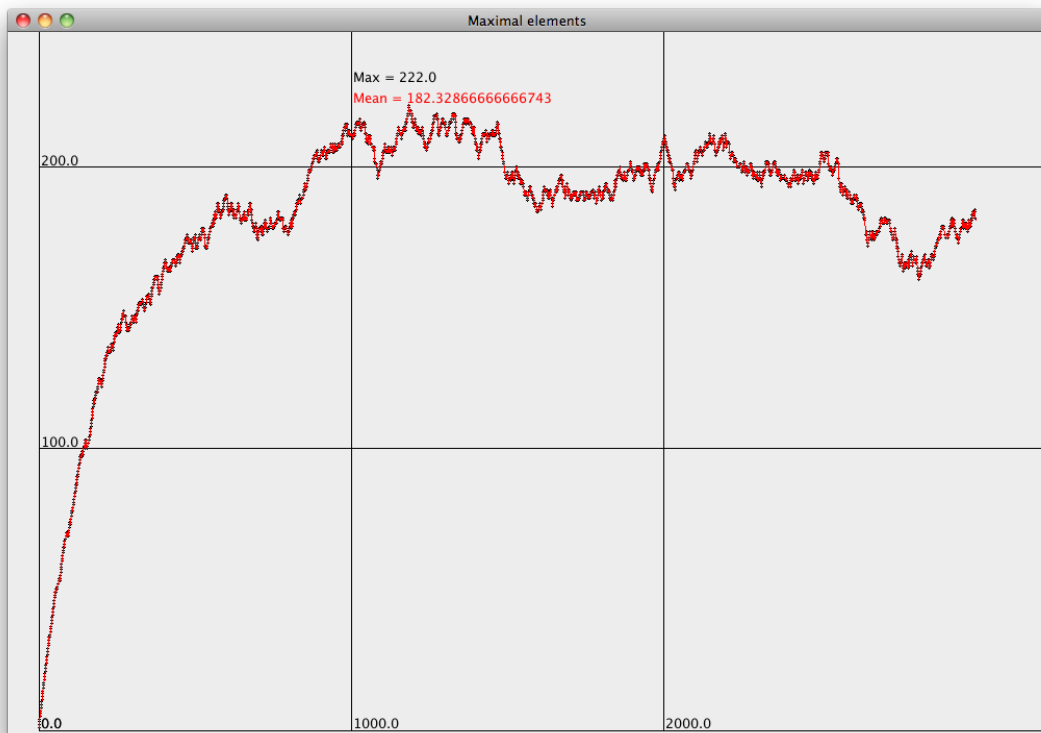


Figure 4.1: Number of maximal elements at birth if k -th element, as function of k . $p = 0.005$, $n = 3000$

4.2 Evolution of the dimension

4.2.1 Dimension as function of p , n

Procedure

We will estimate here the dimension of causal sets using the two techniques explained in Section 3.2: the Myrheim-Meyer and midpoint scaling dimension estimators. Just as we did in Section 3.2, we will look at the impact of both p and n on the dimension.

Let us first see the computing problems we will get for the two estimators. First, in [3], we see that the precision of the Myrheim-Meyer dimension estimator is good for large values of volumes considered. But when p decreases, the volumes obtained also decrease. Hence we need large values of n to compensate. Unfortunately, the dimension estimators are of polynomial complexity as a function of n .

- In the Myrheim-Meyer estimator we need to calculate for every interval ($O(n^2)$) the number of relations ($O(n^2)$) - this gives a $O(n^4)$ algorithm
- In the midpoint scaling estimator we need to calculate for every interval ($O(n^2)$) the corresponding volume ($O(n)$) - this gives a $O(n^3)$ algorithm.

We will get, in our "naive" approach, problems in computing time. Also, we can have problems with space, as all values are stocked in $n \times n$ matrices - the space complexity is quadratic.

If we wanted to enhance the program, we would need:

- a more optimized and larger data structure, for example using SQL;
- a better algorithm to evaluate estimators.

Hence, for computation reasons, we will use here $n \leq 1500$ and $p > 0.01$.

Now before doing the simulation, we need to actually define what we want to compute. In Section 3.2 we have talked about two dimensions, the combinatorial and Hausdorff dimensions, and about two methods to estimate the second one. We didn't say that actually the estimators and the combinatorial dimension are very different. The combinatorial dimension is globally defined - there is a definite integer which describes the whole causal set. On the opposite, the estimators give values which correspond to a local region of the causal set. As the estimator relies on the random nature of the underlying object (we get the estimators when stating that the causal set can be obtained by random sprinkling), we would naturally think about the mean over all considered intervals. However should we consider the small volumes which carry at least a high dispersion and possibly a bias. For example, volumes equal to 3 have dimension $\log_2(3/2)$ according to the midpoint estimator, and intervals of size 4 are of dimension $f^{-1}(1/16)$.

As we have exact theoretical results for combinatorial dimensions, we will compute the Hausdorff dimensions. For these, we will consider only intervals of volume over 50 points, unless stated otherwise. This should cut off the wrong values from small volumes.

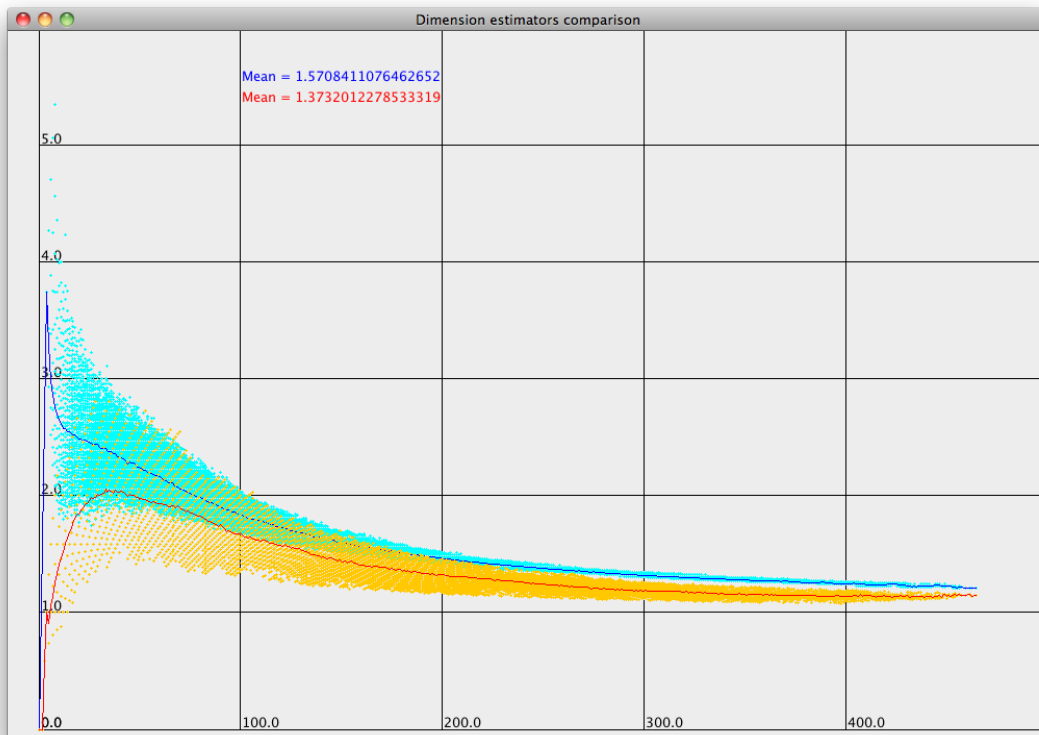


Figure 4.2: Comparison of the Myrheim-Meyer dimension estimator (blue) and the midpoint scaling estimator (red)

Also, we shouldn't consider the locality of the estimators as a problem, but as a force. Using this, we can evaluate the dimension in a given region. Although classical general relativity can have difficulties with a space where dimension is not a constant, with causal sets we can imagine a dimension which does change. Of course, a manifold approximation would be difficult; physically, it is hard to imagine what would a non-integer dimension look like. Anyway, if we forget these problems or assume that they are solved in some way, a varying dimension can be considered. So if we take small enough volumes, we will get a local value of the dimension. For this purpose, we use volumes not greater than 100 elements.

To evaluate the variations of the result as a function of p and n , we will use $p = 0.1, 0.05, 0.01$ and $n = 100, 500, 1000, 1500$. The differences between the values are of only one order, but this has to be enough for now, because of computing limitations.

In Figure 4.2, we compare the results of the Myrheim-Meyer dimension estimator and the midpoint scaling estimator, as a function of the volume of intervals. The results are not equal, but close. In the following, we will concentrate on the midpoint scaling estimator which gives more results due to a lower complexity.

Results

First, let us observe what are the relations between the volume of an interval and its dimension. In a sprinkling of d -dimensional Minkowski space, we observe the figures as in [3]. It is then to be expected that the dimension may vary with the volume, and we can expect that larger volumes give better approximations to the dimension.

The results in Figure 4.3 confirm that the dimension depends on the volume, but instead of seeing a random distribution around some mean value (as we can see in the figures from [3]), we seem to have a dimension distributed around a precise function, growing to a maximum and then decreasing to 1.

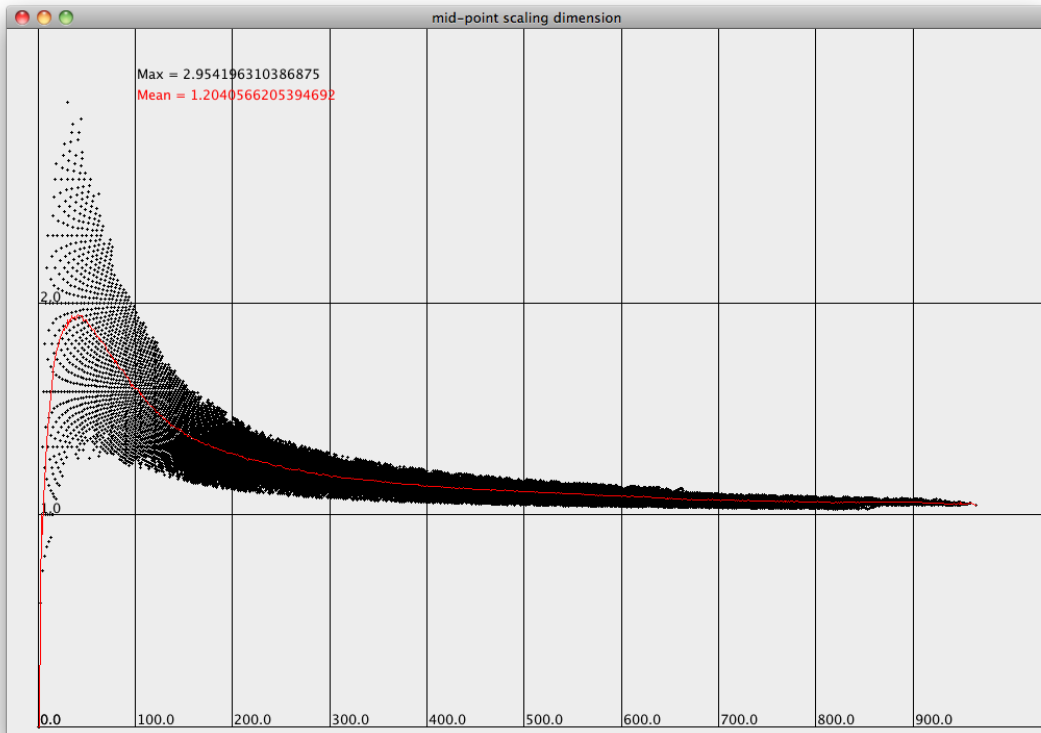
There are several things to comment.

- the dimension of the set goes to 1 when n goes to infinity. Hence, we can extrapolate that if we considered an infinite transitive percolation, the dimension would be roughly 1 - the causal set is basically like a line. This shouldn't come totally as a surprise, as we know that the width of the causal set is finite hence vanishing in comparison with an infinite length. "From far away" the causal set looks like a one-dimensional line, although it isn't exactly one.
- it is unclear whether the initial increase is relevant. For small values, we have large effects due to the integers used in the description. However, the effect can still be seen for larger volumes with lower p .
- we can see that when p decreases, the maximal dimension increases, and the volume for which the maximum is obtained also increases. By extrapolation, we can expect the maximal dimension to be increasing with p .
- for the "local" values of volumes, between 50 and 100, the dimension is also increasing with p
- The value of n doesn't influence the way the curve looks like - every interval is independent of what exists outside of it. Hence n influences the mean value (which goes to 1) and the maximal value only as long as it is small enough for the maximum not to be attained.

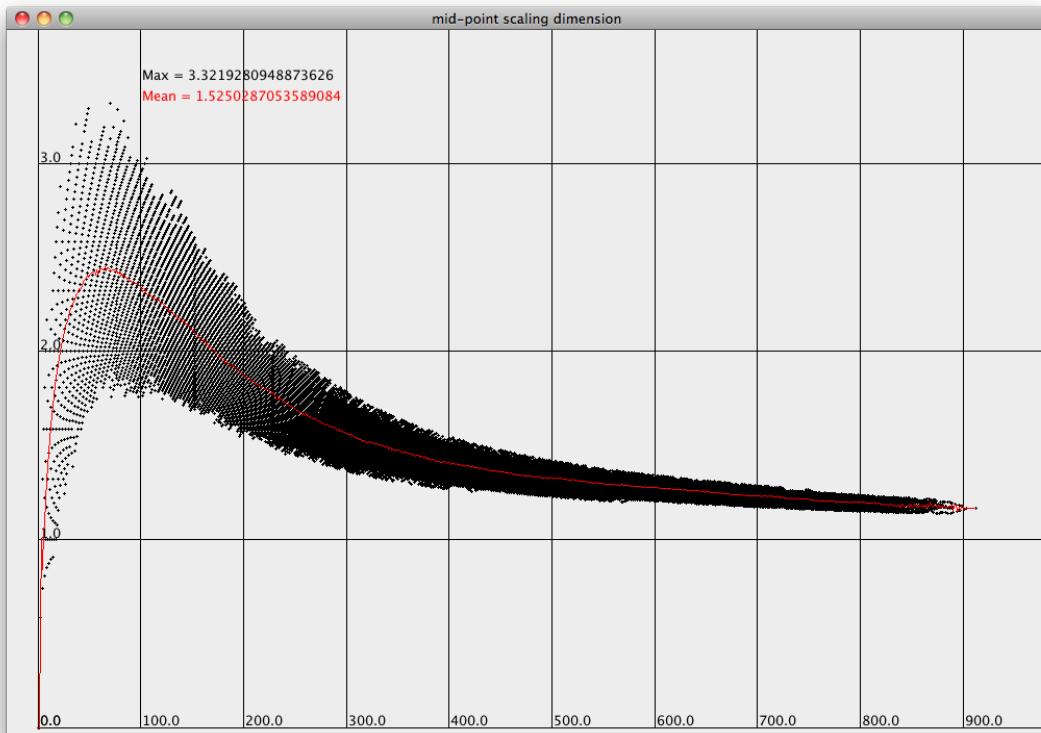
The result is that the causal set has a global "Hausdorff" dimension equal to 1 when big enough. However locally, the dimension can be different. The form of the curve describing dimension shows that the choice of the size of what is local influences the value of local dimensions. This asks for a more thorough inspection of what should be considered as local, but for sake of simplicity, we will stay with the choice of volumes between 50 and 100 elements.

4.2.2 Dimension varying with position

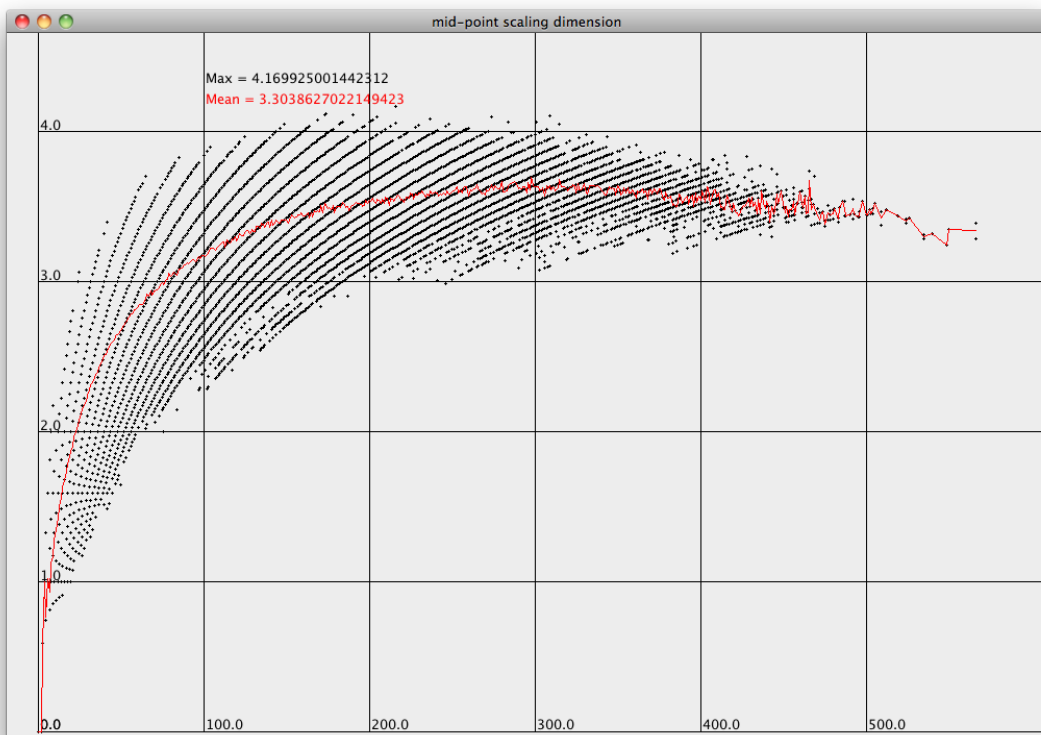
As we have seen in the last section, the dimension estimator's value depends on the volume of the interval considered. This leads to the idea of a local dimension. In the following,



(a) $p = 0.1, n = 1000$



(b) $p = 0.05, n = 1000$



(c) $p = 0.01, n = 1200$

Figure 4.3: midpoint scaling dimension as a function of volume

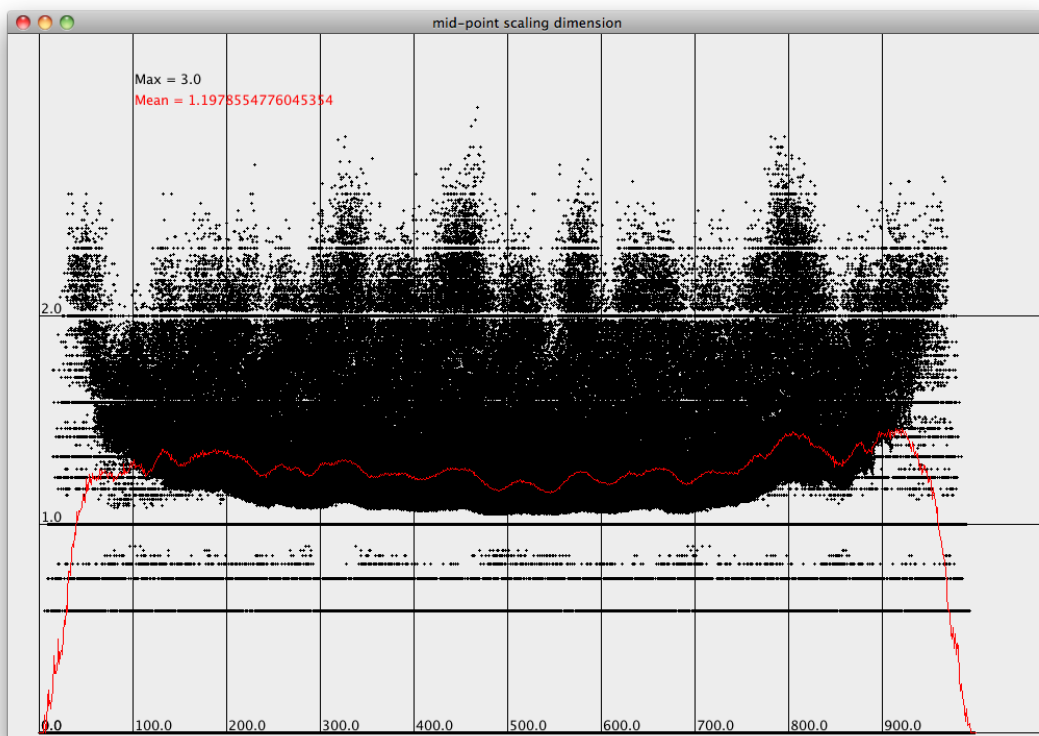
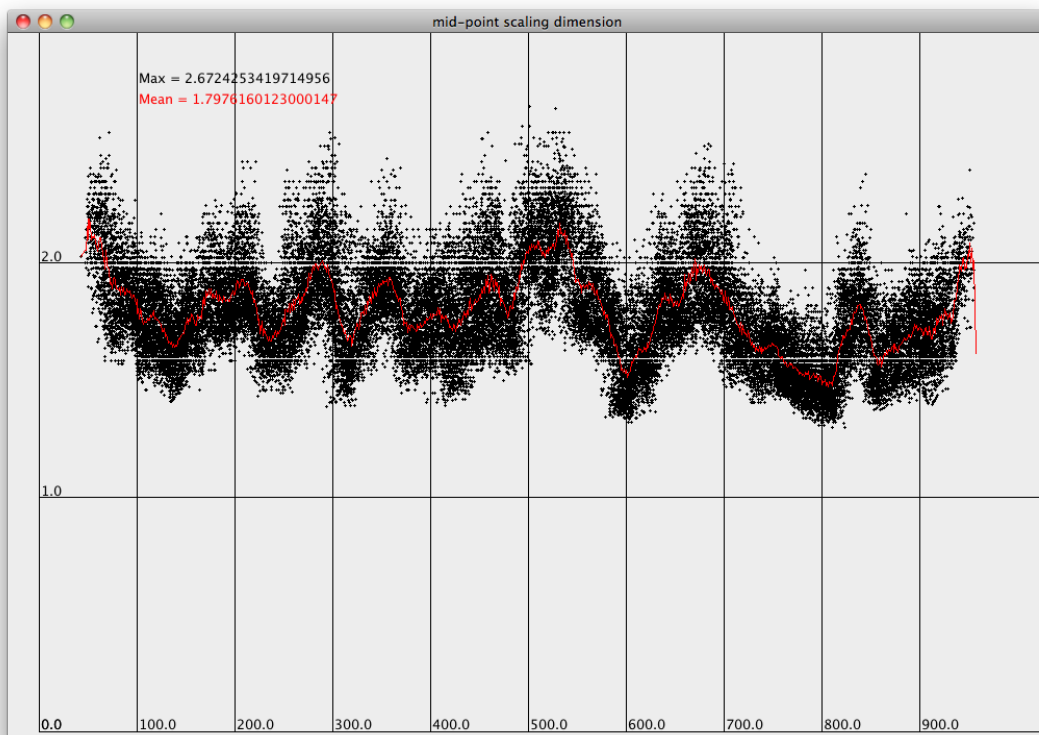
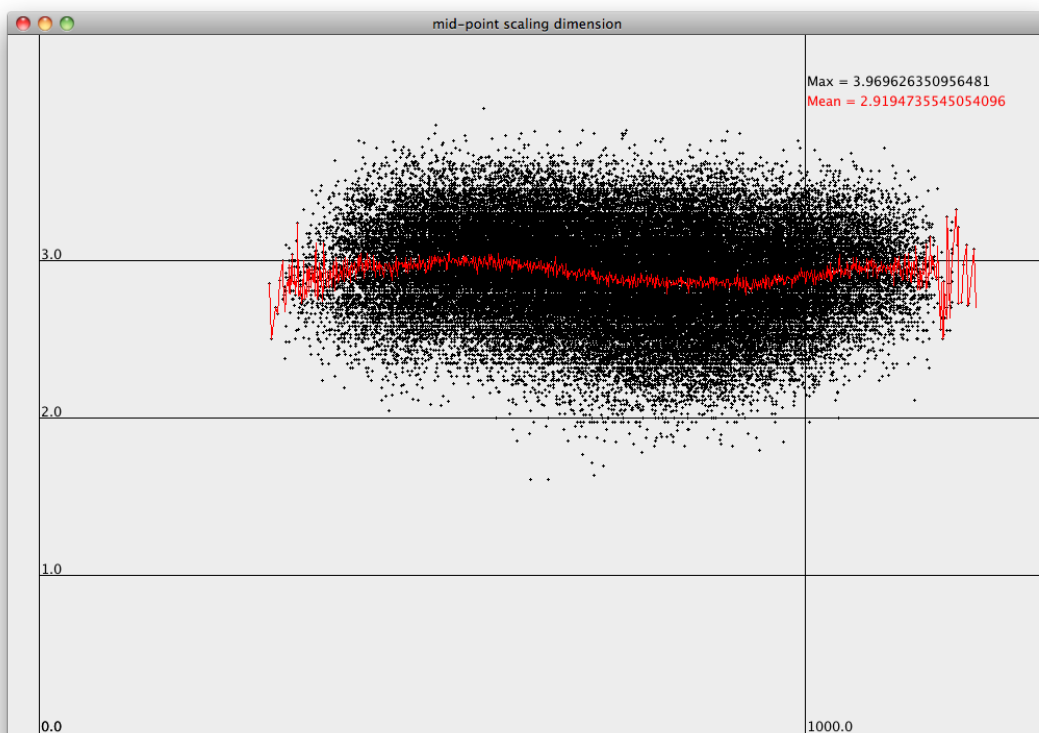


Figure 4.4: Dimension of $[x_i, x_j]$ as function of $\frac{i+j}{2}$. $p = 0.1, n = 1000$

we will try to "localise" every dimension measure. For that, we will place a considered interval $[x_i, x_j]$ at the coordinate $\frac{i+j}{2}$. Now, we can plot the estimation of the dimension versus that coordinate and get the lot on Figure 4.4. This doesn't give much insight on what is happening locally. Now if we select a specific range of volumes, for example between 50 and 100 elements as we have suggested, we have a much different picture. On Figure 4.5a, we can see that the local dimension is randomly distributed around a mean. It seems that the randomness of transitive percolation creates random variations of the "local Hausdorff dimension". In Figure 4.5b the variations can't be seen as they seem to be happening at a larger scale. Indeed, other simulations suggest that the lower p is, the slower the variations of the dimension estimates are.



(a) $p = 0.1, n = 1000$



(b) $p = 0.01, n = 1500$

Figure 4.5: Dimension as function of coordinate, with volume $50 < v < 100$

Chapter 5

Causal set weighting

Sequential growth models are interesting thanks to their simplicity and the remarkable features they present. However, they are not the only candidates as ways of choosing physical causal sets.

We need to explain here more precisely why do we need a dynamics for causal sets. In a way, the simplest idea would be to set a size n and take a generic causal set: perhaps this will look like a manifold. This corresponds to attributing a uniform probability to the set of causets of fixed size. However this fails considerably: according to [4], Kleitman and Rothschild showed that the vast majority of partial orders are completely non manifold like. They have only 3 stages: minimal points, maximal points and a middle level with points unrelated one to another. Hence, it is important to develop a dynamics in which manifold-like causal sets aren't unlikely, to replace the uniform distribution by a more physical probability distribution.

5.1 Sequential growth models as a probability on causal sets

In classical sequential growth models, if we fix an "external time", we fix the number of elements in the causal set. The model gives us probabilities of getting every transition, hence by multiplying we get probabilities of all paths. This sets a probability to every possible causal set of given size according to the specific model.

5.1.1 Probability of a causal set

In the original paper introducing Sequential Growth Models ([4]) one of the conditions for such a model was discrete general covariance. This means that the labelling of the nodes has no physical meaning; for instance, the probability of getting the same through two different processes has to be the same (every process provides a natural labelling). We shall see why this is true.

The probability to follow a specific growth is the product of the probabilities of all transitions, as they are independent processes. At each stage, the denominator is the

same; hence the product of the denominators doesn't depend on the chosen path, and we will write \star instead.

Now the numerator depends on the transition. The value we get is the sum of the t_k which correspond to ancestor sets that lead to the same predecessor set (a predecessor set is obtained with two different ancestor sets as long as they have the same maximal points). Hence, if we consider the transition from C to C' with an additional element in the future of a predecessor set of size k , with m maximal elements, the probability of the transition is

$$\mathbb{P}(C \rightarrow C') = \frac{\sum_{i=0}^{k-m} \binom{k-m}{i} t_{m+i}}{\star}.$$

This depends only on the past of the new element. But as the past is fixed on the moment of birth (no point can be born as a parent of older points), the probability doesn't depend on the order of birth. It depends only on the structure of the past of the point, which is an invariant on all birth orders.

Hence, the product of the numerators doesn't depend on the path either: we can compute the probability of following the path by only looking on the resulting causal set. This gives the probability of a given path. To get the probability of a given causal set, we need to multiply by the number of paths which lead to that result.

5.1.2 In transitive percolation

We could consider transitive percolation as one of many sequential growth models. However, our results from 2.2.1 show that we can equivalently add relations to a set of elements with a fixed probability p and do the transitive closure. Using this, the probability of any causal set is much easier to calculate.

A given causal set is defined by its set of links. Hence, every link had to be added (it couldn't be obtained with transitive closure), and it happened each time with probability p . This gives a probability p^L where L is the number of links. Then, we need to be sure that all non-relations were not added. This gives a probability $q^{\binom{n}{2}-R}$ with $q = 1 - p$ and R the total number of relations, as there are $\binom{n}{2}$ possible relations. Finally, adding any other relation wouldn't change the causal set, as they are added by transitivity anyway.

However this works for a labelled causal set. Labelling a causal set for sequential growth means deciding the order of birth, which means choosing a path. This gives a final factor, the number of paths to get to the causal set. All probabilities being independent by definition, we can multiply all and get:

$$\mathbb{P}(C) = W(C) \cdot p^L \cdot q^{\binom{n}{2}-R}.$$

5.1.3 Automorphisms of causal sets

In the preceding, we have used the number of paths that lead to that causal set $W(C)$. Calculating this is actually not a simple question.

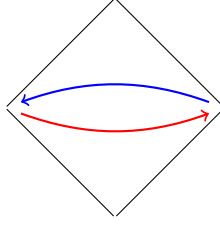


Figure 5.1: Example of automorphism: diamond

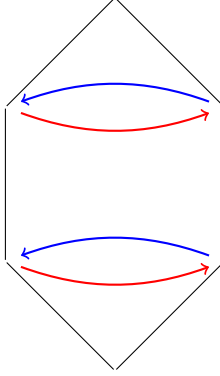


Figure 5.2: Example of automorphism: hexagon

A path corresponds to an order of birth for elements, hence it is a map from the n elements to the set $\{0 \dots n - 1\}$. Since we assumed elements can be born only in the future of existing ones, the order of birth respects the causality relation, i.e. if $i < j$ then $x_i < x_j$. Therefore it is a linear extension of the partial order. Linear extensions have been studied extensively and many results exist on their properties.

However, a path can correspond to many linear extensions. For example, the set of two unrelated points, let's call them a and b , is obtained following a unique path (birth of the first point, then birth of an unrelated point), but has 2 linear extensions ($a < b$ and $b < a$). The relation uses the notion of automorphisms of a causal set. Examples are shown on figures 5.1 and 5.2.

Definition 17. A *causal isomorphism* is a map

$$f : (A, \prec_A) \rightarrow (B, \prec_B)$$

such that

$$\forall x, y \in A, x \prec_A y \Leftrightarrow f(x) \prec_B f(y)$$

Definition 18. An *automorphism* of a causal set (A, \prec) is a causal isomorphism from the causal set to itself (with the same order).

Now paths are linear extensions up to automorphism ([4]). If a causal set has an automorphism, then for every path (which is a linear extension), if we apply a non-identical automorphism, we get a new linear extension. Conversely, if we take two linear extensions

related by automorphism, then they correspond to the same path since paths are defined on poscau which is a set of unlabelled causal sets, hence defined up to automorphism.

We get from this the following result

$$W(C) = \frac{LE(C)}{A(C)}$$

Where $W(C)$ is the number of paths, $LE(C)$ is the number of linear extensions and $A(C)$ is the number of automorphisms.

We can apply this to the following question: what is the most likely result of a sequential growth model after n steps? This suggests that we need to maximize the probability of a path but also maximize the number of paths.

For example, for transitive percolation we need to maximize $W(C) \cdot p^L \cdot q^{\binom{n}{2}-R}$. As $p < 1$ and $q < 1$, we would then look for

- the least links
- the most relations
- the most linear extensions
- the least automorphisms

The conditions seem contradictory, so we have an optimization problem to solve. For example, If we fix the number of links, the maximal number of relations is obtained if we have a chain; however, this has only one linear extension (and one automorphism). If we fix the number of links, the maximal number of linear extensions is obtained by semiorders [18], but they may have less relations than others or many automorphisms.

However, there might be some possible development about the number of automorphisms. Consider the diamond shape from Figure 5.1 and let's evaluate the probability of the appearance of such a shape. We need:

- an element on the left which has its future included in the future of the top one and its past included in the past of the bottom one
- an element on the right which verifies the same and is unrelated to the left one.

Calculations from [8] suggest that the probability of the first to happen is of order p^2 . The probability of the second is of order p^4 as now we need to require that both the right and left element have the same parent and child. Finally, the right element has to be chosen from a pool of p^{-1} points which are unrelated to the left one (simply because if we take a larger set, it is more likely than not to have a relation). This gives a probability of order p^5 for a point to be part of a diamond. This gives around np^5 diamonds in a n -element causal set, so 2^{np^5} automorphisms at least. Then, we should consider more complicated automorphisms, however we can expect them to be far less likely - it seems natural that

the more points will be involved in an automorphism, the less likely it will be. Hence, we can conjecture:

$$\log A(p, n) = Cnp^5(1 + o(1))$$

where $A(p, n)$ is the mean number of automorphisms of a causal set of size n obtained from transitive percolation with parameter p , and C is a constant, when $p \rightarrow 0$.

5.2 Spanning trees and orders

Another idea is to consider the number of spanning trees of a partial order. But first, we need to define a spanning tree for a partial order, as this is an object defined on graphs.

For a graph, a spanning tree is a subgraph which is spanning and which is a tree. It is spanning if it connects all vertices, and it is a tree if it has no cycles.

We could simply consider the partial order as a directed graph and then use the usual definition. This however would give us spanning trees which wouldn't describe at all the structure of the causal set, as a spanning tree wouldn't need to include links. Hence we take a different definition for spanning tree-orders:

Definition 19. A *spanning tree order (STO)* is a suborder which satisfies the following conditions

- *it is connected;*
- *every interval is either null or a chain;*
- *it has a unique minimal element.*

We say that the order is connected if the corresponding undirected graph is connected and spanning. In other words: using paths up or down in the partial order, one can go from any point to any other point. The condition on intervals mean that they are one-dimensional, which among others means that maximal length paths are unique. The last condition is not obvious but is crucial as we will explain.

The idea is that we want to build a bijection between STOs and spanning trees on the graph of links. Take now the 2 dimensional crown: the partial order with 4 points, 2 unrelated on the lower level, both in the past of 2 unrelated points at the top level (Figure 5.3). This is a connected partial order, and every interval is either null or a link. However, the graph of links has an obvious cycle. This problem will not exist for partial orders with a unique minimal point (the crown had 2 minimal points).

Now consider a partial order \prec on a set X and its undirected graph of links G . We will show that the process of taking the undirected graph of links is a bijection between STOs of the partial order and spanning trees of G .

Take an STO \prec_s of the partial order (we can assume that one exists) and consider its undirected graph of links T . We will show that T is a spanning tree. First, it is spanning as \prec_s is connected and we know that every path can be written as a succession of links.

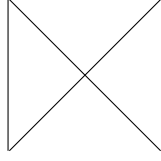


Figure 5.3: Hasse diagram of the 2-dimensional crown

Second, to show that it is a tree, let's assume the contrary: there exists a cycle in T . Let's call the elements of the cycle s_1, \dots, s_k with (s_i, s_{i+1}) and (s_k, s_1) being edges. As edges are links in the STO, we have $s_1 \sim s_2 \sim \dots \sim s_k \sim s_1$ where \sim is either \prec_s or \succ_s . If all are \prec_s or all are \succ_s , then obviously we get a cycle in the STO, which is forbidden by the definition of a partial order. Hence there is at least one s_i such that $s_{i-1} \prec_s s_i \succ_s s_{i+1}$. Now we know that there is a unique minimal element m and we have three possibilities:

- $m = s_{i-1}$ but then $s_{i-1} \prec_s s_{i+1} \prec_s s_i$ hence (s_{i-1}, s_i) is not a link;
- $m = s_{i+1}$ and similarly (s_{i+1}, s_i) is not a link;
- $m \prec_s s_{i-1} \prec_s s_i$ and $m \prec_s s_{i+1} \prec_s s_i$ which shows that the interval $[m, s_i]$ is not a chain.

All cases are impossible, so we conclude that T is a tree - it is then a spanning tree of G .

Now let's fix a labelling of \prec as we will need this to direct the graph. Take the spanning tree T of G , direct it according to the labelling and do the transitive closure to get a partial order \prec_t . Let's show that it is equal to \prec_s . Actually, for two elements x, y , $x \prec_s y$ and $y \prec_t x$ are equivalent to the existence of a sequence of links $\{s_i, s_{i+1}\}$ where the labelling $l(\cdot)$ verifies $l(s_i) < l(s_{i+1})$ and $s_0 = x, s_k = y$. But the set of links and the labelling are common to both, so we conclude that the two orders are the same. This proves the process is bijective.

The immediate result is that the existence of STOs is equivalent to the existence of spanning trees. As spanning trees exist for all connected graphs, we proved the existence of STOs for all connected partial orders. Also, the number of STOs is equal to the number of spanning trees of the graph of links. This gives a practical way to calculate the number of STOs, for example using Kirchoff's tree-matrix theorem [19].

The theorem states that the number of spanning trees is equal to any sub-determinant of the Laplacian matrix of the graph L , which is

$$L = D - A$$

$$D_{ij} = \deg(i) \mathbb{1}_{i=j}$$

$$A_{ij} = \mathbb{1}_{\{i,j\} \in E}$$

Calculating L has then complexity $O(n^2)$. Calculating a determinant can be faster than $O(n^3)$, but let's assume it's $O(n^3)$. Calculating the links of the partial order is also a $O(n^3)$:

for every element ($O(n)$), compare every predecessor to every other ($O(n^2)$). Hence the total algorithm is of order $O(n^3)$ which is acceptable. It is possible to test different causal sets to see which ones have the most spanning trees. This question remains open in general.

Conclusion

It seems quite uncomfortable to believe that spacetime is a causal set and it looks like a 4-dimensional manifold out of pure luck. This motivates the search for causal set dynamics. However perhaps more importantly, if an admissible dynamics was found, i.e. if it produced manifold-like sets with reasonable probability, then analysing the dynamics might show irregularities that experiment could test to see if the causal set hypothesis can be confirmed.

For example, if transitive percolation was seriously proposed as a model for causal sets, then we have seen that it requires a very small parameter p to give the observed width of the universe, but this leads to extremely long interpost distances which basically exclude the possibility of describing a bouncing universe as it is usually presented (growth up to a maximal size followed by contraction). Small values of p might also give large dimensions of the resulting spacetime, although this should be further studied, including the question of what should be considered as local in causal sets. Also, it would be interesting to see what implications on physics would a varying local dimension have.

The global image that arises from transitive percolation is that the resulting causal set is an infinite tube of fixed width (or more exactly of fluctuating width around a mean value) p^{-1} . Every point is quite unlikely to be related to points born shortly before or after itself, but it is almost sure it will be related to points which are far enough from it. Hence every point seems to be surrounded by a "buffer" of basically unrelated points and otherwise is related to all other points. This looks a lot like a semi-order and is certainly not very manifold-like (of dimension ≥ 2).

One might argue that perhaps sequential growth models, a much larger class of models, can yield better results. Unfortunately, this is not the case: in [12], it is proven that continuum limits to classical sequential growth models aren't much different from transitive percolation. In particular, they can't be good approximations to any Minkowski space of dimension larger than 2.

This basically settles the future of sequential growth models as a causal set dynamics. However, let's bear in mind that firstly, it wasn't expected that they would really give the solution to quantum gravity, as there is no quantum in it, and secondly that they were considered because of their interesting features which could be found in future better models. Finally, it shows that there is a need to look for new models, as we did in the last chapter.

Bibliography

- [1] R.D. Sorkin, *Causal Sets: Discrete Gravity*. <http://arxiv.org/abs/gr-qc/0309009v1>, 2002.
- [2] S. W. Hawking, A. R. King, and P. J. McCarthy, *A new topology for curved spacetime which incorporates the causal, differential, and conformal structures*. J. Math. Phys. 17, 174, <http://dx.doi.org.iclibezp1.cc.ic.ac.uk/10.1063/1.522874>, 1976
- [3] David A. Meyer, *The Dimension of Causal Sets*. <http://dspace.mit.edu/handle/1721.1/14328>, 1988.
- [4] D.P. Rideout and R.D. Sorkin, *A Classical Sequential Growth Dynamics for Causal Sets*. <http://arxiv.org/abs/gr-qc/9904062>, 1994.
- [5] T. Britz and P. Cameron, *Partially ordered sets*. <http://www.maths.qmul.ac.uk/~pjc/csgnotes/posets.pdf>, 2001.
- [6] J.H. Kim and B. Pittel, *On Tail Distribution of Interpost Distance*. Journal of Combinatorial Theory, Series B, vol. 80 no.1 pp. 49-56, <http://dx.doi.org/10.1006/jctb.2000.1966>, 2000.
- [7] B. Bollobas and G. Brightwell, *The width of random graph orders*. Mathematical Scientist 20 (1995), no. 2, 69-90.
- [8] B. Bollobas and G. Brightwell, *The structure of random graph orders*. Discrete Mathematics Vol.10 No. 2 pp. 318-335, 1997.
- [9] B. Bollobas and G. Brightwell, *The Dimension of Random Graph Orders*. The Mathematics of Paul Erdos II, pp. 47-68, http://dx.doi.org/10.1007/978-1-4614-7254-4_5, 2013.
- [10] N. Alon, B. Bollobas, G. Brightwell and S. Janson, *Linear extensions of a random partial order*. <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.49.1223>, 1994.
- [11] G. Brightwell, *Photo from "Workshop on Causal Sets"*. http://www.stp.dias.ie/events/2004/causal_sets_photos/WorkshopOnCausalSets-GrahamBrightwell-2.jpg Dublin, 2004.

- [12] G. Brightwell, N. Georgiou, *Continuum limits for classical sequential growth models*. 2009.
- [13] D. Reid, *The Manifold Dimension of a Causal Set: tests in conformally flat spacetimes*. <http://arxiv.org/abs/gr-qc/0207103v2>, 2004.
- [14] S. Felsner, P.C. Fishburn, W.T. Trotter, *Finite three dimensional partial orders which are not sphere orders*. *Discrete Mathematics* 201 101-132, 1999.
- [15] R.D. Sorkin, *Indications of Causal Set Cosmology*. <http://arxiv.org/abs/gr-qc/0003043v3>, 2001.
- [16] M. Ahmed and D.P. Rideout, *Indications of deSitter Spacetime from Classical Sequential Growth Dynamics of Causal Sets*. <http://arxiv.org/abs/0909.4771v2>, 2009.
- [17] D. Dou, *Causal Sets, a Possible Interpretation for the Black Hole Entropy, and Related Topics*. <http://arxiv.org/abs/gr-qc/0106024>, 1999.
- [18] P.C. Fishburn and W.T. Trotter, *Linear extensions of semiorders: A maximization problem*. *Discrete Mathematics* 103 (1992) pp. 25-40.
- [19] F. Harary and E.M. Palmer, *Graphical Enumeration*. Academic Press, 1973.